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FINITE DOMINATION AND NOVIKOV RINGS

ANDREW RANICKI

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0. INTRODUCTION

In order to distinguish between the combinatorial properties of finite simplicial complexes and the topology of compact polyhedra and compact manifolds it is necessary to consider infinite simplicial complexes, non-compact polyhedra, open manifolds, and algebraic K- and L-theory. The classic cases are the Milnor Hauptvermutung counterexamples of non-combinatorial homeomorphisms of compact polyhedra, the proof by Novikov of the topological invariance of the rational Pontrjagin classes, and the structure theory of Kirby and Siebenmann for high-dimensional compact topological manifolds. The open manifolds arise geometrically as tame ends: in the applications it is necessary to close them. The obstruction theory for closing tame ends of open manifolds is also the obstruction theory for deciding if a finitely dominated space is homotopy equivalent to a finite CW complex.

Definition 1. A topological space X is *finitely dominated* if it is a homotopy retract of a finite CW complex, i.e. if there exist a finite CW complex K, maps $f: X \to K$, $g: K \to X$ and a homotopy $gf \simeq 1: X \to X$.

The Wall [20] finiteness obstruction $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of a finitely dominated space X is such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex.

The finite domination properties of infinite cyclic covers of finite CW complexes are of particular interest. In fact, every finitely dominated space X is homotopy equivalent to an infinite cyclic cover of a finite CW complex (see Section 4 for a proof).

Here is how infinite cyclic covers of finite CW complexes arise in geometric topology. A tame end ε of an open n-dimensional manifold W has a finitely dominated neighbourhood $\bar{V} \subset W$ which is an infinite cyclic cover of a compact n-dimensional manifold V, the "wrapping up" of ε with $V \times \mathbb{R} \cong \bar{V} \times S^1 \subset W \times S^1$. It is possible to express $\bar{V}$ as a union $\bar{V}^+ \cup \bar{V}^-$ with $\bar{V}^+ \cap \bar{V}^-$ a compact $(n-1)$-dimensional manifold and $\bar{V}^+$, $\bar{V}^-$ finitely dominated, with $\pi_1(\bar{V}^+) = \pi_1(\bar{V}^-) = \pi_1(\bar{V}) = \pi_1(\varepsilon)$. The end obstruction of Siebenmann [14] is the finiteness obstruction $[\varepsilon] = [\bar{V}^+] \in \tilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon)])$, with $[\varepsilon] = 0$ if (and for $n \geq 6$ only if) the tame end can be closed.

In this paper the Novikov rings of formal power series will be used to obtain a homological characterization of finite domination for an infinite cyclic cover of a finite CW complex. In [13] this characterization will be applied to the study of fibre bundles over S^1, fibred knots, the bordism of diffeomorphisms and open book decompositions. In [5] this characterization will be related to the topological notions of forward and reverse tameness for open manifolds considered by Quinn [11].

Definition 2. The *Novikov rings* of a ring A are the completions $A((z))$, $A((z^{-1}))$ of the Laurent polynomial extension $A[z, z^{-1}]$ given by

$$A((z)) = \left\{ \sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{ j \leq 0 \,|\, a_j \neq 0 \in A \} \text{ finite} \right\}$$

$$A((z^{-1})) = \left\{ \sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{ j \geq 0 \,|\, a_j \neq 0 \in A \} \text{ finite} \right\}$$

with intersection

$$A((z)) \cap A((z^{-1})) = A[z, z^{-1}] = \left\{ \sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{ j \in \mathbb{Z} \,|\, a_j \neq 0 \in A \} \text{ finite} \right\}.$$

Traditional Morse theory deals with $\mathbb{R}$-valued functions of compact manifolds. Novikov [9] suggested the use of these rings in counting the critical points of S^1-valued Morse functions on compact manifolds M, initially in the case $\pi_1(M) = \mathbb{Z}$, $A = \mathbb{Z}$. Pazhitnov [10] showed how to apply the Novikov rings with $A = \mathbb{Z}[\pi]$ to the Morse theory of S^1-valued Morse functions on arbitrary compact manifolds M with $\pi_1(M) = \pi \times \mathbb{Z}$. The Novikov rings also appear in the Morse-theoretic chain complex construction of Floer homology by Hofer and Salamon [4].

Definition 3. The Λ-*coefficient homology* of a connected CW complex X is

$$H_*(X; \Lambda) = H_*(\Lambda \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{X}))$$

with Λ a $\mathbb{Z}[\pi_1(X)]$-module and $C(\tilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex of the universal cover $\tilde{X}$.

Algebraic K-theory decides if a finitely dominated space is homotopy equivalent to a finite CW complex. Homology with coefficients in the Novikov rings decides if an infinite cyclic cover of a finite CW complex is finitely dominated.

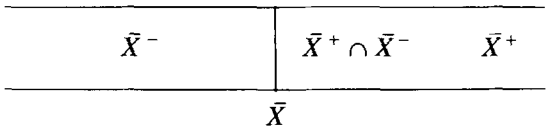
THEOREM 1. *Let X be a finite CW complex with universal cover $\tilde{X}$ and fundamental group $\pi_1(X) = \pi \times \mathbb{Z}$, so that $\mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi][z, z^{-1}]$. The infinite cyclic cover $\bar{X} = \tilde{X}/\pi$ is finitely dominated if and only if X is $\mathbb{Z}[\pi]((z))$- and $\mathbb{Z}[\pi]((z^{-1}))$-acyclic*

$$H_*(X; \mathbb{Z}[\pi]((z))) = H_*(X; \mathbb{Z}[\pi]((z^{-1}))) = 0.$$

Theorem 1 is proved in Section 5 by an application to the cellular chain complex $C(\tilde{X})$ and the group ring $A = \mathbb{Z}[\pi]$ of the corresponding characterization of finite domination for a finite finitely generated (f.g.) free $A[z, z^{-1}]$-module chain complex C valid for arbitrary A. By definition, C is A-finitely dominated if and only if it is A-module chain equivalent to a finite f.g. projective A-module chain complex. It is known from the work of Wall [20] that $\bar{X}$ is finitely dominated if and only if $C(\tilde{X})$ is $\mathbb{Z}[\pi]$-finitely dominated. It is proved in Section 5 that C is A-finitely dominated if and only if

$$H_*(A((z)) \otimes_{A[z,z^{-1}]} C) = H_*(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) = 0.$$

However, the proof is best understood in terms of the topology of infinite cyclic covers of compact spaces. The two conditions in Theorem 1 arise as follows: by the CW analogue of manifold transversality it is possible to decompose the infinite cyclic cover $\bar{X}$ of the finite CW complex X as a union $\bar{X} = \bar{X}^+ \cup \bar{X}^-$ of infinite subcomplexes $\bar{X}^+, \bar{X}^- \subset \bar{X}$ with $\bar{X}^+ \cap \bar{X}^-$ finite and $\pi_1(\bar{X}^+) = \pi_1(\bar{X}^-) = \pi_1(\bar{X}) = \pi$.



The cover $\bar{X}$ is finitely dominated if and only if both $\bar{X}^+$ and $\bar{X}^-$ are finitely dominated. It turns out that $H_*(X; \mathbb{Z}[\pi]((z))) = 0$ if and only if $\bar{X}^-$ is finitely dominated, $H_*(X; \mathbb{Z}[\pi]((z^{-1}))) = 0$ if and only if $\bar{X}^+$ is finitely dominated. See Section 6 for the connection with the algebraic chain complex theory of tame ends developed in Hughes and Ranicki [5].

Example. The universal cover of S^1 is $\tilde{S}^1 = \mathbb{R}$ with

$$C(\mathbb{R}): \ldots \to 0 \to \mathbb{Z}[z, z^{-1}] \xrightarrow{1-z} \mathbb{Z}[z, z^{-1}]$$

and

$$H_*(S^1; \mathbb{Z}[z, z^{-1}]) = H_*(1 - z : \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]) = \mathbb{Z}$$
$$H_*(S^1; \mathbb{Z}((z))) = H_*(1 - z : \mathbb{Z}((z)) \to \mathbb{Z}((z))) = 0$$
$$H_*(S^1; \mathbb{Z}((z^{-1}))) = H_*(1 - z : \mathbb{Z}((z^{-1})) \to \mathbb{Z}((z^{-1}))) = 0$$

since $1 - z \in \mathbb{Z}[z, z^{-1}]$ becomes a unit in $\mathbb{Z}((z))$ and $\mathbb{Z}((z^{-1}))$ with

$$(1 - z)^{-1} = \sum_{i=0}^{\infty} z^i \in \mathbb{Z}((z)), \qquad (1 - z)^{-1} = -\sum_{i=-\infty}^{-1} z^i \in \mathbb{Z}((z^{-1})).$$

A compact n-dimensional manifold M with a finitely dominated infinite cyclic cover $\bar{M}$ is a potential fibre bundle over S^1. The manifold M is a fibre bundle over S^1 if and only if $\bar{M} = c^*\mathbb{R}$ for a Morse function $c: M \to S^1$ with no critical points. Farrell [3] and Siebenmann [17] formulated the obstruction to M being a fibre bundle over S^1 as an element $\Phi(M) \in Wh(\pi_1(M))$, with $\Phi(M) = 0$ if (and for $n \geq 6$ only if) M fibres over S^1.

For any group π define the abelian group

$$Wh((\pi \times \mathbb{Z})) = K_1(\mathbb{Z}[\pi]((z)))/\{\pm \pi \times \mathbb{Z}, 1 + z\mathbb{Z}[\pi][[z]]\}$$

by analogy with the Whitehead group $Wh(\pi) = K_1(\mathbb{Z}[\pi])/\{\pm \pi\}$, with

$$\pm \pi \times \mathbb{Z} = \{\pm g z^j \mid g \in \pi, j \in \mathbb{Z}\}$$
$$1 + z\mathbb{Z}[\pi][[z]] = \left\{ \sum_{j=0}^{\infty} a_j z^j \mid a_j \in \mathbb{Z}[\pi], a_0 = 1 \right\}$$

the subgroups of $K_1(\mathbb{Z}[\pi]((z)))$ generated by trivial units in $\mathbb{Z}[\pi]((z))$.

Pazhitnov [10] conjectured that for a compact manifold M with $\pi_1(M) = \pi \times \mathbb{Z}$ the infinite cyclic cover $\bar{M} = \tilde{M}/\pi$ is finitely dominated if and only if $H_*(M; \mathbb{Z}[\pi]((z))) = 0$. The S^1-valued Morse theory of Novikov [9] was developed further in [10], allowing the

$\mathbb{Z}[\pi]((z))$-coefficient Reidemeister torsion

$$\tau(M; \mathbb{Z}[\pi]((z))) = \tau(\mathbb{Z}[\pi]((z)) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\tilde{M})) \in Wh((\pi \times \mathbb{Z}))$$

to be regarded as a fibering obstruction. The following corollary verifies this conjecture, and clarifies the precise relationship between the two fibering obstructions. It will be proved in detail in [13].

COROLLARY. *Let M be a compact n-dimensional manifold with $\pi_1(M) = \pi \times \mathbb{Z}$. The infinite cyclic cover $\bar{M} = \tilde{M}/\pi$ of M is finitely dominated if and only if $H_*(M; \mathbb{Z}[\pi]((z))) = 0$, in which case the natural map $Wh(\pi \times \mathbb{Z}) \to Wh((\pi \times \mathbb{Z}))$ sends the Farrell–Siebenmann fibering obstruction $\Phi(M) \in Wh(\pi \times \mathbb{Z})$ to the Pazhitnor Fibering construction $\tau(M; \mathbb{Z}[\pi]((z))) \in Wh((\pi \times \mathbb{Z}))$, and $\Phi(M) = 0$ if and only if $\tau(M; \mathbb{Z}[\pi]((z))) = 0$. Thus, if $\bar{M}$ is finitely dominated $\tau(M; \mathbb{Z}[\pi]((z))) = 0$ if (and for $n \geq 6$ only if) fibres over S^1, as proved in [10] using S^1-valued Morse theory.*

Idea of Proof. The $\mathbb{Z}[\pi]((z))$-coefficient homology is such that $H_*(M; \mathbb{Z}[\pi]((z))) = 0$ if and only if $H_*(M; \mathbb{Z}[\pi]((z^{-1}))) = 0$, by Poincaré duality and the universal coefficient theorem. The natural map $Wh(\pi \times \mathbb{Z}) \to Wh((\pi \times \mathbb{Z}))$ is the projection sending one of the $\widetilde{\text{Nil}}$-factors in the Bass decomposition of $Wh(\pi \times \mathbb{Z})$ to 0:

$$\begin{aligned} Wh(\pi \times \mathbb{Z}) &= Wh(\pi) \oplus \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \\ &\to Wh((\pi \times \mathbb{Z})) = Wh(\pi) \oplus \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \end{aligned}$$

and the two $\widetilde{\text{Nil}}$-components of $\Phi(M) \in Wh(\pi \times \mathbb{Z})$ are dual to each other. □

The fundamental group of a connected infinite cycle cover $\bar{X}$ of a connected space X is the α-twisted extension of $\pi_1(\bar{X})$ by the infinite cyclic group $\mathbb{Z}$,

$$\pi_1(X) = \pi_1(\bar{X}) \times_\alpha \mathbb{Z}$$

with $\alpha = \zeta_* : \pi_1(\bar{X}) \to \pi_1(\bar{X})$ the automorphism induced (up to inner automorphisms) by a generating covering translation $\zeta : \bar{X} \to \bar{X}$. For the sake of simplicity, we shall only be concerned with the untwisted case $\alpha = 1$, $\pi_1(X) = \pi_1(\bar{X}) \times \mathbb{Z}$, but all the results obtained here extend to the general case with arbitrary α, subject to the additional hypothesis that $\pi_1(\bar{X})$ is finitely presented.

1. ALGEBRAIC TRANSVERSALITY

The homological characterization of finite domination is an application of an algebraic theory of transversality for chain complexes over polynomial extensions, which mimics the existence of compact fundamental domains for infinite cyclic covers of compact manifolds.

The infinite cyclic covers $\bar{X}$ of a space X are classified by the homotopy classes of maps $c : X \to S^1$, with

$$\bar{X} = \{(x, t) \in X \times \mathbb{R} \mid c(x) = [t] \in S^1 = \mathbb{R}/\mathbb{Z}\}.$$

The connected covers $\bar{X}$ of a connected space X correspond to maps c with $c_* : \pi_1(X) \to \pi_1(S^1) = \mathbb{Z}$ onto, in which case the fundamental group exact sequence of the fibration $\bar{X} \to X \xrightarrow{c} S^1$ expresses $\pi_1(X)$ as an extension of $\pi_1(\bar{X})$ by $\mathbb{Z}$:

$$\{1\} \to \pi_1(\bar{X}) \to \pi_1(X) \to \mathbb{Z} \to \{1\}.$$

The connected covers $\bar{X}$ are thus classified by the normal subgroups $\pi \subset \pi_1(X)$ with infinite cyclic quotient $\pi_1(X)/\pi = \mathbb{Z}$, such that

$$\bar{X} = \tilde{X}/\pi, \qquad \pi_1(\bar{X}) = \pi$$

with $\tilde{X}$ the universal cover of X. As stipulated in the Introduction it will be assumed that the generating covering translation

$$\zeta : \bar{X} \to \bar{X} \qquad x \to zx$$

induces an inner automorphism $\zeta_* : \pi_1(\bar{X}) = \pi \to \pi$, so that

$$\pi_1(X) = \pi \times \mathbb{Z}, \qquad \mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi][z, z^{-1}].$$

Geometric Transversality. *Let M be a compact n-dimensional manifold with $\pi_1(M) = \pi \times \mathbb{Z}$ and infinite cyclic cover $\bar{M} = \tilde{M}/\pi$. The classifying map $c : M \to S^1$ can be made transverse regular at a point $* \in S^1$, and cutting M along a codimension 1 submanifold $N = c^{-1}(*) \subset M$ gives a compact fundamental domain $(M_N; N, \zeta N)$ with*

$$M = M_N \cup N \times [0, 1], \qquad \bar{M} = \bigcup_{j=-\infty}^{\infty} \zeta^j M_N, \qquad \pi_1(M_N) = \pi_1(N) = \pi.$$

The two inclusions $f, g : N \to M_N$ determine an exact sequence of $\mathbb{Z}[\pi][z, z^{-1}]$-module chain complexes

$$0 \to C(\tilde{N})[z, z^{-1}] \xrightarrow{f - zg} C(\tilde{M}_N)[z, z^{-1}] \to C(\tilde{M}) \to 0$$

with $\tilde{M}$, $\tilde{N}$, $\tilde{M}_N$ the universal covers of M, N, M_N.

Algerbraic Transversality (Waldhausen [19] and Ranicki [12, 8.12]). *Every finite based f.g. free $A[z, z^{-1}]$-module chain complex C is such that there is defined a simple exact sequence*

$$0 \to D[z, z^{-1}] \xrightarrow{f - zg} E[z, z^{-1}] \to C \to 0$$

for some finite based f.g. free A-module chain complexes D, E and A-module chain maps $f, g : D \to E$. In fact, E can be chosen to be the f.g. free A-module subcomplex of C generated by base elements of the type $z^j b$ for the base elements b of C, and

$$f : D = E \cap z^{-1}E \to E; \quad x \to zx$$

$$g : D = E \cap z^{-1}E \to E; \quad x \to x.$$

The proof of Theorem 2 will make use of the following consequence of algebraic transversality.

Proposition 1. *For any finite f.g. free $A[z, z^{-1}]$-module chain complex C there exist a finite f.g. free $A[z]$-module subcomplex $C^+ \subset C$ and a finite f.g. free $A[z^{-1}]$-module subcomplex $C^- \subset C$ such that $C^+ \cap C^-$ is a finite f.g. free A-module chain complex and*

$$A[z, z^{-1}] \otimes_{A[z]} C^+ = A[z, z^{-1}] \otimes_{A[z^{-1}]} C^- = C$$

with a Mayer–Vietoris exact sequence

$$0 \to C^+ \cap C^- \to C^+ \oplus C^- \to C \to 0.$$

Proof. Choose bases for C, let D, E, f, g be as in the statement of algebraic transversality above, and set

$$C^+ = \mathrm{im}(E[z] \to C), \qquad C^- = \mathrm{im}(E[z^{-1}] \to C)$$

so that $C^+ \cap C^- = E$. □

2. THE MAPPING TORUS

The mapping torus is a homotopy model for a space with an infinite cyclic cover.

Definition 4. The *mapping torus* of a self-map $f: X \to X$ is the identification space

$$T(f) = X \times [0, 1]/\{(x, 0) = (f(x), 1) \mid x \in X\}$$

with canonical infinite cyclic cover the two-sided mapping telescope

$$\bar{T}(f) = \left(\coprod_{n=-\infty}^{\infty} X \times [0, 1] \times \{n\} \right) \Big/ \{(x, 0, n) = (f(x), 1, n+1) \mid x \in X\}.$$

An infinite cyclic cover of a space X corresponds to a mapping torus in the homotopy type of X.

PROPOSITION 2. *If $p: \bar{X} \to X$ is the projection of an infinite cyclic cover with covering translation $\zeta: \bar{X} \to \bar{X}$ then*

$$q: T(\zeta) \to X; \qquad (\bar{x}, t) \to p(\bar{x})$$

is a homotopy equivalence.

'WHITEHEAD LEMMA' FOR MAPPING TORI (Mather [6]). *For any maps $f: X \to Y$, $g: Y \to X$ there are defined inverse homotopy equivalences*

$$T(gf) \to T(fg); \qquad (x, s) \to (f(x), s)$$

$$T(fg) \to T(gf); \qquad (y, t) \to (g(y), t).$$

3. HOMOTOPY FINITENESS

Finitely dominated spaces are topological analogues of f.g. projective modules, which are the direct summands of f.g. free modules. The difference between the homotopy types of finite and finitely dominated CW complexes is precisely the difference between f.g. projective and f.g. free modules.

An infinite cyclic cover $\bar{X}$ of a finite CW complex X is an infinite CW complex; in general, it is not even homotopy equivalent to a finite CW complex.

A finite f.g. free $A[z, z^{-1}]$-module chain complex C is not finitely generated over A, and is not in general chain equivalent to a finite f.g. projective A-module chain complex.

Definition 5. A topological space X is *homotopy finite* if it is homotopy equivalent to a finite CW complex.

Example. If $X = K \times S^1$ for a finite CW complex K then the infinite cyclic cover $\bar{X} = K \times \mathbb{R} \simeq K$ is homotopy finite.

PROPOSITION 3. *A topological space X is finitely dominated if and only if $X \times S^1$ is homotopy finite.*

Proof. Given a finite domination $(K, f: X \to K, g: K \to X, gf \simeq 1: X \to X)$ apply the Mather lemma to obtain a homotopy equivalence

$$X \times S^1 \simeq T(gf) \simeq T(fg) = \text{finite } CW \text{ complex.}$$

The converse is trivial. □

Finite domination and homotopy finiteness are detected on the chain level by the following.

WALL FINITENESS OBSTRUCTION [20]. *A connected CW complex X is finitely dominated (resp. homotopy finite) if and only if the fundamental group $\pi_1(X)$ is finitely presented and the cellular free $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$ is chain equivalent to a finite f.g. projective (resp. free) $\mathbb{Z}[\pi_1(X)]$-module chain complex. If X is finitely dominated the reduced projective class of any finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex P in the chain homotopy type of $C(\tilde{X})$:*

$$[X] = \sum_{r=0}^{\infty} (-)^r [P_r] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)]) = \frac{\{\text{f.g. projective } \mathbb{Z}[\pi_1(X)]\text{-modules}\}}{\{\text{f.g. free } \mathbb{Z}[\pi_1(X)]\text{-modules}\}}$$

is the finiteness obstruction, such that $[X] = 0$ if and only if X is homotopy finite.

SIEBENMANN END OBSTRUCTION [14]. *An open n-dimensional manifold W with one tame end ε admits a finitely dominated cocompact closed neighbourhood of the end $X \subseteq W$ with $\pi_1(X) = \pi_1(\varepsilon)$. The end obstruction of ε is defined to be the finiteness obstruction of any such X,*

$$[\varepsilon] = [X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon)]).$$

The end obstruction is such that $[\varepsilon] = 0 \in \tilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon)])$ if (and for $n \geq 6$ only if) W is the interior of a compact n-dimensional manifold.

It is clear that every homotopy finite space is finitely dominated. In the simply connected case $\pi_1(X) = \{1\}$ every finitely dominated space X is homotopy finite, since $\tilde{K}_0(\mathbb{Z}) = 0$. A simply connected space is finitely dominated if and only if it is homotopy finite, and there is a standard homological criterion for homotopy finiteness in this case.

PROPOSITION 4 (Milnor (unpublished) and Spanier [18, Exercise 8.G.5]). *A simply connected CW complex X is homotopy finite if and only if the homology $H_*(X)$ is a f.g. $\mathbb{Z}$-module.*

Likewise for nilpotent spaces, we have the following proposition.

PROPOSITION 5 (Mislin [8]). *A nilpotent space X is finitely dominated if and only if the homology $H_*(X)$ is a f.g. $\mathbb{Z}$-module.*

4. BANDS

Definition 6 (Siebenmann [16]). A *band* is a finite CW complex X with a finitely dominated infinite cyclic cover $\bar{X}$.

PROPOSITION 6. *Every finitely dominated CW complex X is homotopy equivalent to the infinite cyclic cover $\bar{Y}$ of a band Y.*

Proof. For any finite domination $(K, f: X \to K, g: K \to X, gf \simeq 1: X \to X)$ there is defined a homotopy equivalence

$$X \times S^1 \simeq T(gf) \simeq T(fg) = \text{finite } CW \text{ complex} = Y$$

with $X \simeq X \times \mathbb{R} \simeq \bar{T}(gf) \simeq \bar{T}(fg) \simeq \bar{Y}$. Y is a band. □

Example (Siebenmann [15]). A non-compact n-dimensional manifold W with one tame end ε has an open neighbourhood $\bar{V} \subseteq W$ which is the infinite cyclic cover of a compact n-dimensional manifold band V, the *wrapping up* of the end, with $\bar{V} = \bar{V}^+ \cup \bar{V}^-$ such that

$$\pi_1(\varepsilon) = \pi_1(V^+) = \pi_1(\bar{V})$$

$$[\varepsilon] = [\bar{V}^+] \in \tilde{K}(\mathbb{Z}[\pi_1(\varepsilon)]).$$

The idea is to use tameness and a proper Morse function $W \to \mathbb{R}^+$ to lift $\mathbb{R}^+ \to \mathbb{R}^+$; $x \to x+1$ to a shift map $T: W \to W$ which is a covering translation on an open neighbourhood of the end. See [5] for a more detailed treatment of wrapping up.

5. ALGEBRAIC FINITE DOMINATION

Definition 7. A finite f.g. free $A[z, z^{-1}]$-module chain complex C is *A-finitely dominated* if it is A-module chain equivalent to a finite chain complex of f.g. projective A-modules, in which case it is a *chain complex band.*

Example (Cayley–Hamilton, Milnor [7]). If F is a field then a finite f.g. free $F[z, z^{-1}]$-module chain complex C is F-finitely dominated if and only if

$$H_*(F(z) \otimes_{F[z, z^{-1}]} C) = 0$$

with $F(z)$ the function field of F (= the quotient field of $F[z]$).

Proof. If C is F-finitely dominated the homology $H_*(C)$ is a finite-dimensional F-vector space, and the characteristic polynomial of the "monodromy"

$$\zeta: H_*(C) \to H_*(C); \quad x \to zx$$

is a non-zero polynomial

$$p(z) = \det(z - \zeta: H_*(C)[z] \to H_*(C)[z]) \in F[z]$$

such that $p(z)H_*(C) = 0$.

Conversely, if $H_*(F(z) \otimes_{F[z, z^{-1}]} C) = 0$ there exists a non-zero polynomial $p(z) \in F[z]$ such that $p(z)H_*(C) = 0$. The ring $F[z, z^{-1}]$ is noetherian, so that $H_*(C)$ is a f.g. $F[z, z^{-1}]$-module, hence a f.g. $F[z, z^{-1}]/(p(z))$-module, and hence a f.g. F-module. □

A connected finite CW complex X with $\pi_1(X) = \pi \times \mathbb{Z}$ is a band if and only if the cellular $\mathbb{Z}[\pi][z, z^{-1}]$-module chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$ is a chain complex band, by Wall's chain level criterion for finite domination.

THEOREM 2. *A finite f.g. free $A[z, z^{-1}]$-module chain complex C is A-finitely dominated if and only if*

$$H_*(A((z)) \otimes_{A[z,z^{-1}]} C) = H_*(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) = 0.$$

Proof. Let $i: A \to A[z, z^{-1}]$ be the inclusion, and let

$$i_! = \text{induction}: \{A\text{-modules}\} \to \{A[z, z^{-1}]\text{-modules}\}$$

$$i^! = \text{restriction}: \{A[z, z^{-1}]\text{-modules}\} \to \{A\text{-modules}\}.$$

Given an A-module P let

$$P[z, z^{-1}] = i_! P = A[z, z^{-1}] \otimes_A P, \quad P((z)) = A((z)) \otimes_A P$$

be the induced $A[z, z^{-1}]$- and $A((z))$-modules. For any A-modules P, Q there is a natural injection

$$\mathrm{Hom}_{A((z))}(P((z)), Q((z))) \to \mathrm{Hom}_A(P, Q)((z)).$$

If $h: P \to P$ is an automorphism of a f.g. projective A-module P then $z - h: P((z)) \to P((z))$ is an automorphism, with inverse

$$(z - h)^{-1} = (-h)^{-1} \sum_{j=0}^{\infty} (h^{-1}z)^j : P((z)) \to P((z)).$$

Warning: this is definitely false if P is not f.g. projective, e.g. if

$$h: P = i^! A[z, z^{-1}] \to P; \qquad \sum_{j=-\infty}^{\infty} a_j z^j \to \sum_{j=-\infty}^{\infty} a_j z^{j+1}$$

in which case there is defined an exact sequence

$$0 \to A[z, z^{-1}] \to P((z)) \xrightarrow{z-h} P((z)) \to 0$$

with

$$A[z, z^{-1}] \to P((z)); \qquad \sum_{j=-\infty}^{\infty} a_j z^j \to \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h^{-k}(a_j) z^{j+k}.$$

A finite f.g. free $A[z, z^{-1}]$-module chain complex C is chain equivalent to the algebraic mapping torus

$$T(\zeta) = \mathscr{C}(z - \zeta : i_! i^! C \to i_! i^! C)$$

of the A-module automorphism

$$\zeta : i^! C \to i^! C; \qquad x \to zx$$

of the infinitely generated free A-module chain complex $i^! C$. If C is A-finitely dominated then $i^! C$ is A-moduled chain equivalent to a finite f.g. projective A-module chain complex P with an A-module chain equivalence $h: P \to P$ such that C is $A[z, z^{-1}]$-module chain equivalent to $T(h)$. Since P is f.g. projective it is possible to identify

$$\mathrm{Hom}_{A((z))}(P((z)), P((z))) = \mathrm{Hom}_A(P, P)((z)).$$

The $A[z, z^{-1}]$-module chain map $z - h: P[z, z^{-1}] \to P[z, z^{-1}]$ induces an $A((z))$-module chain equivalence $z - h: P((z)) \to P((z))$, so that

$$A((z)) \otimes_{A[z,z^{-1}]} C \simeq A((z)) \otimes_{A[z,z^{-1}]} T(h) \simeq 0.$$

Similarly for $A((z^{-1}))$.

For the converse use the cartesian squares of rings

$$\begin{array}{ccc} A[z] & \to & A[z,z^{-1}] \\ \downarrow & & \downarrow \\ A[[z]] & \to & A((z)) \end{array} \qquad \begin{array}{ccc} A[z^{-1}] & \to & A[z,z^{-1}] \\ \downarrow & & \downarrow \\ A[[z^{-1}]] & \to & A((z^{-1})). \end{array}$$

By Proposition 1 for every finite f.g. free $A[z,z^{-1}]$-module chain complex C there exist a finite f.g. free $A[z]$-module subcomplex $C^+ \subset C$ and a finite f.g. free $A[z^{-1}]$-module subcomplex $C^- \subset C$ with

$$C = A[z,z^{-1}] \otimes_{A[z]} C^+ = A[z,z^{-1}] \otimes_{A[z^{-1}]} C^-$$

and such that $C^+ \cap C^-$ is a finite f.g. free A-module chain complex, with a Mayer–Vietoris exact sequence

$$0 \to C^+ \cap C^- \to C^+ \oplus C^- \to C \to 0.$$

The chain complex C is A-finitely dominated if and only if C^+ and C^- are A-finitely dominated. We shall show that $A((z^{-1})) \otimes_{A[z,z^{-1}]} C \simeq 0$ implies that C^+ is A-finitely dominated. The f.g. free $A[z^{-1}]$-module chain complex C^- fits into an exact sequence of $A[z^{-1}]$-module chain complexes

$$0 \to C^- \xrightarrow{i} C^-[z,z^{-1}] \oplus C^-[[z^{-1}]] \to C^-((z^{-1})) \to 0$$

with

$$\begin{aligned} C^-[z,z^{-1}] &= A[z,z^{-1}] \otimes_{A[z^{-1}]} C^- = C \\ C^-[[z^{-1}]] &= A[[z^{-1}]] \otimes_{A[z^{-1}]} C^- \\ C^-((z^{-1})) &= A((z^{-1})) \otimes_{A[z^{-1}]} C^- = A((z^{-1})) \otimes_{A[z,z^{-1}]} C. \end{aligned}$$

By hypothesis $H_*(C^-((z^{-1}))) = 0$, so i is an $A[z^{-1}]$-module chain equivalence. Let

$$j : C^+ \cap C^- \to C^+ \oplus C^-[[z^{-1}]]$$

be the A-module chain map defined by inclusions in each component. The algebraic mapping cones of i and j are chain equivalent A-module chain complexes, since $C/C^- \cong C^+/(C^+ \cap C^-)$. But i is a chain equivalence, so that j is also a chain equivalence. Since $C^+ \cap C^-$ is a finite f.g. free A-module chain complex this shows that both C^+ and $C^-[[z^{-1}]]$ are A-finitely dominated. Similarly, $A((z)) \otimes_{A[z,z^{-1}]} C \simeq 0$ implies that C^- is A-finitely dominated. □

Corollary. *A connected finite CW complex X with universal cover $\tilde{X}$ and fundamental group $\pi_1(X) = \pi \times \mathbb{Z}$ is a band if and only if*

$$H_*(\mathbb{Z}[\pi]((z)) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\tilde{X})) = H_*(\mathbb{Z}[\pi]((z^{-1})) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\tilde{X})) = 0. \qquad □$$

Example. For any f.g. projective A-module $P = \mathrm{im}(p = p^2 : A^k \to A^k)$ the 1-dimensional $A[z,z^{-1}]$-module chain complex

$$\cdots \to 0 \to P[z,z^{-1}] \xrightarrow{1-z} P[z,z^{-1}]$$

is chain equivalent to the chain complex band

$$C: \cdots \to 0 \to C_1 = A[z,z^{-1}]^k \xrightarrow{d = 1 - pz} C_0 = P[z,z^{-1}]^k$$

with

$$H_0(C) = P \text{ (z acting by the identity)}$$
$$H_0(A((z)) \otimes_{A[z,z^{-1}]} C) = H_0(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) = 0$$
$$(1 \otimes d)^{-1} = 1 + p \sum_{j=1}^{\infty} z^j : A((z)) \otimes_{A[z,z^{-1}]} C_0 \to A((z)) \otimes_{A[z,z^{-1}]} C_1$$
$$(1 \otimes d)^{-1} = 1 - p \sum_{j=-\infty}^{0} z^j : A((z^{-1})) \otimes_{A[z,z^{-1}]} C_0 \to A((z^{-1})) \otimes_{A[z,z^{-1}]} C_1.$$

This is an algebraic analogue of the Mather construction of a CW band X in the homotopy type of $Y \times S^1$ for a finitely dominated CW complex Y. □

The following is an example of an infinite cyclic cover of a finite CW complex which is not finitely dominated.

Example (Milnor [7]). If $X = S^1 \vee S^2$ then $\bar{X} = \mathbb{R} \cup \mathbb{Z} \times S^2$ is not finitely dominated, since $\bar{X}$ is simply connected and $H_2(\bar{X}) = \mathbb{Z}[z, z^{-1}]$ is not a f.g. $\mathbb{Z}$-module. The cellular $\mathbb{Z}[z, z^{-1}]$-module chain complex is

$$C(\bar{X}): \ \dots \to 0 \to \mathbb{Z}[z, z^{-1}] \xrightarrow{0} \mathbb{Z}[z, z^{-1}] \xrightarrow{1-z} \mathbb{Z}[z, z^{-1}]$$

and

$$H_2(X; \mathbb{Z}((z))) = \mathbb{Z}((z)), \qquad H_2(X; \mathbb{Z}((z^{-1}))) = \mathbb{Z}((z^{-1})).$$

Remark. Bieri and Eckmann [1] and Brown [2] obtained a necessary and sufficient homological criterion for a projective A-module chain complex C to be finitely dominated, namely that the functor

$$H_*(C, -): \{\text{right } A\text{-modules}\} \to \{\text{graded } \mathbb{Z}\text{-modules}\}$$
$$M \to H_*(C; M) = H_*(M \otimes_A C)$$

preserve products. (I am indebted to C.T.C. Wall for these references.) A f.g. free $A[z, z^{-1}]$-module chain complex is a free A-module chain complex, but it is not clear how this criterion for finite domination is related to the criterion provided by Theorem 2.

6. ENDS OF COMPLEXES

The proof of Theorem 2 will now be related to the chain complex properties of tame ends, as promised in the Introduction.

Given a space W let

$$W^\infty = W \cup \{\infty\}$$

be the one-point compactification.

Definition 7 (Quinn [11]). (i) The *homotopy link of* ∞ *in* W^∞ is the space $e(W)$ of proper paths

$$\omega: (0, 1] \to W$$

or equivalently maps

$$\omega:([0,1],\{0\})\to(W^\infty,\{\infty\})$$

such that $\omega^{-1}(\infty)=\{0\}$.

(ii) A non-compact space W is *tame at* ∞ if there exists a closed cocompact subspace $V\subseteq W$ such that the inclusion $V\times\{1\}\to W$ extends to a proper map $q:V\times(0,1]\to W$, or equivalently a map

$$\bar{q}:(V\times(0,1])^\infty = V^\infty\wedge[0,1]\to W^\infty$$

such that $(\bar{q})^{-1}(\infty)=\infty$.

If W is tame at ∞ the homotopy link is such that there is defined a homotopy pushout

$$\begin{array}{ccc} e(W) & \to & \{\infty\} \\ {\scriptstyle p_W}\downarrow & & \downarrow \\ W & \xrightarrow{i} & W^\infty \end{array}$$

with $i: W\to W^\infty$ the inclusion and

$$p_W:e(W)\to W;\qquad \omega\to\omega(1).$$

For such W the homology groups of $e(W)$ fit into an exact sequence

$$\cdots\to H_n(e(W))\to H_n(W)\to H_n^{lf}(W)\to H_{n-1}(e(W))\to\cdots$$

with $H_*^{lf}(W)=H_*(W^\infty,\{\infty\})$ the locally finite homology groups of W.

The homotopy link is a homotopy theoretic model for the topology of an end of a non-compact space which is tame at ∞. An infinite cyclic cover $W=\bar{X}$ of a connected finite CW complex X has two ends. To each path $\omega\in e(W)$ assign a sign $\pm$ according as to which end contains $\omega((0,1])$, and lift $p_W:e(W)\to W$ to a map

$$\bar{p}_W:e(W)\to W\times\{\pm\}=W\sqcup W;\qquad \omega\to(\omega(1),\pm).$$

If W is finitely dominated then W is tame at ∞, and $\bar{p}_W$ is a homotopy equivalence.

The combinatorial and chain level properties of the homotopy link construction are investigated in [5], including the definition of the *end complex* $e(C)$ of a based free A-module chain complex C. If

$$C_r=\sum_{I_r}A\ (r\in\mathbb{Z})$$

then the A-module chain complex $e(C)$ is defined to be the algebraic mapping cone of the inclusion $i:C\to C^{lf}$

$$e(C)=\mathscr{C}(i:C\to C^{lf})_{*+1}$$

with C^{lf} the locally finite chain complex given by

$$C_r^{lf}=\prod_{I_r}A\ (r\in\mathbb{Z}).$$

It is shown in [5] that if W is tame at ∞, and $\tilde{W}$ is the universal cover of W, then the pullback cover $\widetilde{e(W)}$ of $e(W)$ is such that

$$H_*(\widetilde{e(W)})=H_*(e(C(\tilde{W})))$$

with $\tilde{p}_W:H_*(\widetilde{e(W)})\to H_*(\tilde{W})$ induced by the projection $e(C(\tilde{W}))\to C(\tilde{W})$.

Let X be a finite CW complex with an infinite cyclic cover $W = \bar{X}$ classified by

$$c = \text{projection}: \pi_1(X) = \pi \times \mathbb{Z} \to \mathbb{Z}$$

with a finite fundamental domain $(X_Y;\ Y, \zeta Y)$ such that

$$W = W^+ \cup W^- = \bigcup_{j=-\infty}^{\infty} \zeta^j(X_Y;\ Y, \zeta Y)$$

$$W^+ = \bigcup_{j=0}^{\infty} \zeta^j X_Y, \quad W^- = \bigcup_{j=-\infty}^{-1} \zeta^j X_Y, \quad W^+ \cap W^- = Y$$

$$\pi_1(W) = \pi_1(W^+) = \pi_1(W^-) = \pi_1(Y) = \pi_1(X_Y) = \pi.$$

(Every finite CW complex X with $\pi_1(X) = \pi \times \mathbb{Z}$ is simple homotopy equivalent to one of this type — e.g. a closed regular neighbourhood of X in some Euclidean space. See Ranicki [12, 8.15] for a combinatorial transversality construction.) Then W is finitely dominated if and only if W^+ and W^- are finitely dominated, in which case the composites

$$e(W^+) \xrightarrow{p_{W^+}} W^+ \to W$$

$$e(W^-) \xrightarrow{p_{W^-}} W^- \to W$$

$$e(W^+) \sqcup e(W^-) \to e(W)$$

are homotopy equivalences. Let $\tilde{Y}, \tilde{X}_Y, \tilde{W}$ be the universal covers of Y, X_Y, W, so that

$$\tilde{W} = \tilde{W}^+ \cup_{\tilde{Y}} \tilde{W}^- = \bigcup_{j=-\infty}^{\infty} \zeta^j(\tilde{X}_Y;\ \tilde{Y}, \zeta \tilde{Y})$$

$$\tilde{W}^+ = \bigcup_{j=0}^{\infty} \zeta^j \tilde{X}_Y, \qquad \tilde{W}^- = \bigcup_{j=-\infty}^{-1} \zeta^j \tilde{X}_Y, \qquad \tilde{W}^+ \cap \tilde{W}^- = \tilde{Y}.$$

Let $A = \mathbb{Z}[\pi]$, so that $A[z, z^{-1}] = \mathbb{Z}[\pi \times \mathbb{Z}]$. The cellular chain complexes

$$C(\tilde{W}^+) = C^+, \qquad C(\tilde{W}^-) = C^-$$

are such that C^+ (resp. C^-) is a based f.g. free $A[z]$- (resp. $A[z^{-1}]$-) module chain complex. The cellular chain complex of $\tilde{W}$ is given by

$$C(\tilde{W}) = C = A[z, z^{-1}] \otimes_{A[z]} C^+ = A[z, z^{-1}] \otimes_{A[z^{-1}]} C^-$$

and the locally π-finite cellular chain complexes of $\tilde{W}^+$, $\tilde{W}^-$ are given by

$$C(\tilde{W}^+)^{lf} = A[[z]] \otimes_{A[z]} C^+, \qquad C(\tilde{W}^-)^{lf} = A[[z^{-1}]] \otimes_{A[z^{-1}]} C^-.$$

The end complexes of C^+, C^- are given by

$$e(C^+) = \mathscr{C}(i: C^+ \to A[[z]] \otimes_{A[z]} C^+)_{*+1}$$

$$e(C^-) = \mathscr{C}(i: C^- \to A[[z^{-1}]] \otimes_{A[z^{-1}]} C^-)_{*+1}.$$

The condition $H_*(A((z^{-1})) \otimes_{A[z]} C^+) = 0$ for C^+ to be A-finitely dominated is just that the composite

$$e(C^-) \xrightarrow{\text{proj.}} C^- \xrightarrow{\text{incl.}} C$$

be a homology equivalence. The two conditions of Theorem 2 for a finite f.g. free $A[z, z^{-1}]$-module chain complex C to be A-finitely dominated

$$H_*(A((z)) \otimes_{A[z,z^{-1}]} C) = H_*(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) = 0$$

are just that both the chain maps $e(C^+) \to C$, $e(C^-) \to C$ be homology equivalences, for any Mayer–Vietoris exact sequence

$$0 \to C^+ \cap C^- \to C^+ \oplus C^- \to C \to 0$$

given by algebraic transversality, with $C^+ \cap C^-$ a finite f.g. free A-module chain complex.

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REFERENCES

1. R. BIERI and B. ECKMANN: Finiteness properties of duality groups, *Comm. Math. Helv.* **49** (1974), 74–83.
2. K. BROWN: Homological criteria for finiteness, *Comm. Math. Helv.* **50** (1975), 129–135.
3. F. T. FARRELL: The obstruction of fibering a manifold over a circle, *Indiana Univ. J.* **21** (1971), 315–346.
4. H. HOFER and D. SALAMON: Floer homology and Novikov rings, to appear in *Floer memorial volume.*
5. B. HUGHES and A. RANICKI: Ends of complexes, preprint.
6. M. MATHER: Counting homotopy types of manifolds, *Topology* **4** (1965), 93–94.
7. J. MILNOR: Infinite cyclic covers, *Proc. 1967 Conf. on the topology of manifolds*, Prindle, Weber and Schmidt, Boston, (1968), 115–133.
8. G. MISLIN: Wall's obstruction for nilpotent spaces, *Ann. Math.* **103** (1976) 547–556.
9. S. P. NOVIKOV: The Hamiltonian formalism and a multivalued analogue of Morse theory, *Uspekhi Mat. Nauk* **37** (1982), 3–49. (English: *Russ. Math. Surv.* **37** (1982), 1–56.)
10. A. V. PAZHITNOV: Surgery on the Novikov complex, preprint (1992).
11. F. QUINN: Homotopically stratified sets, *J. AMS* **1** (1988), 441–499.
12. A. A. RANICKI: *Lower K- and L-theory*, LMS Lecture Notes 178, Cambridge Univ. Press, Cambridge (1992).
13. A. A. RANICKI: The algebraic theory of bands, preprint.
14. L. SIEBENMANN: The obstruction to finding the boundary of an open manifold, Ph. D. thesis (1965).
15. L. SIEBENMANN: The structure of tame ends, Notices AMS 66T-G7 (1966), 861.
16. L. SIEBENMANN: A torsion invariant for bands, Notices AMS 68T-G7 (1968), 811.
17. L. SIEBENMANN: A total Whitehead torsion obstruction to fibering over the circle, *Comm. Math. Helv.* **45** (1972), 1–48.
18. E. SPANIER: *Algebraic Topology*, McGraw-Hill, New York (1966).
19. F. WALDHAUSEN: Algebraic K-theory of generalized free products, *Ann. Math.* **108** (1978), 135–256.
20. C. T. C. WALL: Finiteness conditions for CW complexes, *Ann. Math.* **81** (1965), 56–69.

Department of Mathematics and Statistics
University of Edinburgh
Edinburgh EH9 3JZ
UK